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Entropic Uncertainty Relation

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Abstract

Shannon's entropy is used for the formulation of the uncertainty relation. Two concrete examples are solved: the case of dynamically commuting continuous observables, and the case of the $h/2$ spin projections.

1. *Introduction*

It is recognised that the mathematical base of the uncertainty principle rests upon the statistical dependence of non-commuting quantum observables, known as physical random variables (Majerník, 1970).

This dependence is mathematically given by a set of conditional probabilities. In quantum mechanics the dependence is usually known, being determined by transformation rules between different representations of a state vector (unitary transformations). Such knowledge allows us to express one of the measures of statistical dependence by means of the so-called uncertainty relation. For the formulation of the uncertainty relation statistical dispersions are used (Heisenberg, 1930; Jackiw, 1968). We shall call this formulation the standard one, in order to distinguish the other possible expressions of the statistical dependence among the observables.

There are no deeper mathematical or physical reasons for expressing the statistical dependences in the standard form. The use of dispersions of observables can, perhaps, be justified by tradition (an estimation of the accuracy of a measurement or an estimation of fluctuations etc.) and also by the fact that the probability distributions of values of sufficiently punctually determined quantum observables are similar to the Gaussian probability distribution which is completely given by the average value—the mathematical expectation and by the dispersion.

In order to determine the statistical dependence of observables we can use other and often more general and convenient measures of the random variables constructed in probability theory. These measures can be the functions of the random variables and elements of a corresponding probability distribution, for example, the statistical moments or the functions of elements of the probability distributions only—so-called measures of the probability uncertainty, for example, Shannon's entropy.

As to the former measures, if they are used, for instance, even higher statistical moments centred according to the average value (with respect to the possibility of the odd moments being negative and with respect to the fact that the average values have an exceptional meaning in quantum mechanics) than we obtain by means of the variational method analogous to that developed by Jackiw (1968), awkward operational equations which attain their simplest form in the standard case. The present author believes that knowledge of the solution of a particular non-standard equation can give nothing new. Something new could perhaps be gained by knowledge of the solutions of all equations but, as we know, at the present time such solutions are not available. Thus if we want to use statistical moments to express measures of the uncertainty it is more convenient to use dispersions. There are, however, objections to the use of the dispersions as measure of the uncertainty of the probability distributions. If we have, for instance, probability distributions not of Gaussian form or even with some different maximums the dispersion becomes an unsuitable measure of the uncertainty. This can be illustrated by an instructive and simple mathematical example. Let us consider a probability distribution determined by its density:

$$P(x)=\begin{cases}0 & -\infty<x<a\\ 1/N & a\leqslant x\leqslant b\\ 0 & b<x<c\\ 1/N & c\leqslant x\leqslant d\\ 0 & d<x<+\infty\end{cases} \tag{1.1}$$

where $N=d-c+b-a$. We define some new parameters

$$L_1=b-a \qquad L_2=d-c$$

determining the length of the regions of the non-zero probability density,

$$T=(a+b+c+d)/4$$

characterising symmetry of their locations, and

$$L=(d+c-b-a)/2$$

referring to a mutual distance between the centres of the regions of the non-zero probability density. Then the dispersion of a random variable x is

$$\langle (X-\langle X\rangle)^2\rangle = L^2\frac{L_1L_2}{(L_1+L_2)^2}+\tfrac{1}{12}(L_1^2-L_1L_2+L_2^2) \tag{1.2a}$$

As we see result (1.2a) contains an unsuitable dependence on the parameter L—it diverges as L^2 if $L \to \infty$.

Another disadvantageous feature of the dispersion appears in the case of discrete probability distributions. The uncertainty product can attain zero minimum even when one of the distributions is not absolutely localised, i.e. if the value of one of the observables is not precisely determined. The uncertainty or the statistical dependence is just characterised by the lower bound of the uncertainty product and in the above-mentained case this characterisation becomes meaningless.

We shall try to remove the former difficulties by means of the probability entropy of an observable. The probability entropy can be defined as a natural measure of the uncertainty of a probability distribution (Jaynes, 1957; Fadejev, 1967). Let us recall some definitions and features of this entropy (Jaynes, 1957; Fano, 1959; Renyi, 1962). If the values $x \in \chi$ of the random variable X are realised with probability $P(x)$ then the entropy of the probability distribution $P(x)$ is

$$W(X) = -\sum_{x \in \chi} P(x) \ln P(x) \tag{1.3}$$

One of the characteristic properties of the entropy consists in the fact that the larger the uncertainty of a probability distribution the larger is its entropy. For the continuous observable we take the entropy as

$$P(X) = -\int_{\chi} P(x) \ln P(x)\, dx \tag{1.4}$$

where $P(x)$ is the probability density. Although the entropy (1.4) has not all the characteristics of the entropy (1.3) it can still be used as a (differential or relative) measure of the uncertainty. The value of the entropy (1.4) of an absolutely localised probability distribution, described by the probability density $P(x) = \delta(x - x')$, by definition, can be equal to $-\infty$ ($W(X) = -\infty$). We note that the entropy of this distribution is not defined because the $\ln \delta(x)$ is not defined (Bremermann, 1965). In reality, there are no absolutely localised (continuous) probability distributions but only distributions very similar to them. These distributions, as well as the distribution $P(x) = \delta(x - x')$, can have the arbitrary accuracy approximated by the Gaussian one and, limiting the parameter determining the dispersion to zero, we obtain $W(X) = -\infty$.

In order to make a comparison we give the entropy (1.4) of the distribution (1.1)

$$W(X) = \ln (L_1 + L_2)$$

which is independent of the parameter L.

After some preliminary notes, we devote ourselves to the study of the uncertainty principle as formulated by means of the probability entropy. At first we formulate the entropic uncertainty relation in the general case. We then concentrate upon two concrete examples: the case of the canonically conjugate observables in the Weyl sense (Garrison & Wong, 1970) (having a continuous spectrum of eigenvalues) and the case of spin variables (having a discrete spectrum of eigenvalues).

2. *Entropic Uncertainty Relation*

For the formulation of the uncertainty relation we shall use the so-called *a priori* entropies. Let us consider two observables F and G. We denote the entropies of the observables as $W(F)$ and $W(G)$. We shall study the sum

$$W = W(F) + W(G) \tag{2.1}$$

i.e. the sum of the uncertainties. As in the standard case, we are interested in the lower bound of the uncertainty sum (2.1) and in the state belonging to this bound—the optimal state. The lower bound of the sum (2.1) can never be less than

$$W_{\min} = W_{F,\min} + W_{G,\min} \tag{2.2}$$

where $W_{F,\min}$ and $W_{G,\min}$ are the minimal values of the *a priori* entropies (in the case of a continuous spectrum they are equal to $-\infty$ and in the case of a discrete spectrum they are equal to zero). The uncertainty sum (2.1) can attain its minimum value (2.2) if the observables F and G are statistically independent of each other, i.e. if their operators commute with each other. However, the value (2.2) can also be attained if the observables F and G are statistically dependent and there exists a state vector which is an eigenvector of both operators F and G simultaneously. This is impossible if the commutator $[F, G]$ is a (non-zero) c-number. Since the entropy of an absolutely localised probability distribution is always less than the entropy of any other distribution the case referred to in the introduction can never occur, namely, the case of the discrete spectrum where the dispersion of one observable is zero and the dispersion of a second one is non-zero, therefore the uncertainty product equals zero. If the probability distributions are continuous, the sum (2.1) could have a drawback if a state vector exists which is an eigenvector, for instance of the operator $\hat{F}$, i.e. $W(F) = -\infty$ and $|W(G)| < \infty$. Then the sum of uncertainties (2.1) would attain its minimum $W = -\infty$.

We shall call the sum of the uncertainties (2.1) completed by a really attainable minimum the entropic uncertainty relation.

We now derive equations for the searching minimum of the sum of the uncertainties. For the remainder of this section the symbol Σ means summation or integration in the case of discrete or continuous distributions, respectively. The eigenvectors and the eigenvalues of the operator $\hat{F}$ we denote by $|f$ and f, respectively. Similarly for the $\hat{G}$ operator, we have $|g$ and g. The state of a quantum mechanical system is described by the state operator

$$\hat{\rho} = \sum_{\alpha} |\alpha\rangle \omega_{\alpha} < \alpha| \tag{2.3a}$$

$$\mathrm{Tr}\,(\hat{\rho}) = 1 \tag{2.3b}$$

The probability (in the case of a continuous spectrum the probability density) that the observable F has a value f is

$$P(f) = \langle f | \hat{\rho} | f \rangle = \sum_{\alpha} \langle f | \alpha \rangle \, \omega_{\alpha} \, \langle \alpha | f \rangle \tag{2.4a}$$

and similarly

$$P(g) = \langle g | \hat{\rho} | g \rangle = \sum_{\alpha} \langle g | \alpha \rangle \, \omega \, \langle \alpha | g \rangle \tag{2.4b}$$

Inserting equations (2.4) into equation (2.1) and using the variational method with condition (2.3) we find

$$-\sum_{f} \langle f | \alpha \rangle \langle \alpha | f \rangle \ln \langle f | \hat{\rho} | f \rangle - \sum_{g} \langle g | \alpha \rangle \langle \alpha | g \rangle \ln \langle g | \hat{\rho} | g \rangle - W = 0 \tag{2.5a}$$

$$-\sum_{f} | f \rangle \omega_{\alpha} \langle \alpha | f \rangle \ln \langle f | \hat{\rho} | f \rangle - \sum_{g} | g \rangle \omega_{\alpha} \langle \alpha | g \rangle \ln \langle g | \hat{\rho} | g \rangle - W \omega_{\alpha} | \alpha \rangle = 0 \tag{2.5b}$$

These equations can be written in the following form

$$\frac{\partial W}{\partial \omega_{\alpha}} = W \tag{2.6a}$$

$$\frac{\delta W}{\delta \langle \alpha |} = W \omega_{\alpha} | \alpha \rangle \tag{2.6b}$$

The calculations are usually made in some actual representation. If we choose, for instance, the f-representation then the state vectors $| \beta \rangle$ must be changed by

$$| \beta \rangle = \sum_{f} | f \rangle \langle f | \beta \rangle$$

and the variational variables are ω_{α} and $\langle \alpha | f \rangle$. Equation (2.5a) does not change and equation (2.5b) is now

$$-\omega_{\alpha} \langle f | \alpha \rangle \ln \langle f | \hat{\rho} | f \rangle - \sum_{g} \langle f | g \rangle \omega_{\alpha} \langle g | \alpha \rangle \ln \langle g | \hat{\rho} | g \rangle - W \omega_{\alpha} \langle f | \alpha \rangle = 0 \tag{2.7}$$

Next, for the sake of simplicity, we confine ourselves to the pure states.

3. *Entropic Uncertainty Relation for the Canonically Conjugated Observables in the Weyl Sense*

Canonically conjugated observables in the Weyl sense are, for instance, the coordinate x and the momentum p of a free particle in one dimension. As is known, all canonically conjugate observables in the Weyl sense are equivalent to them (Garrison & Wong, 1970; Von Neumann, 1932).

We use equation (2.7) and calculations are made in the x-representation, i.e. $\hat{F} = x$ and $\hat{G} = \hat{p} = -ihd/dx$. Since only the pure states are studied, the index α is left. We insert the following expressions into equation (2.7):

$$\langle x \mid \alpha \rangle = \varphi_\alpha(x) = \varphi(x)$$
$$\langle p \mid \alpha \rangle = \Psi_\alpha(p) = \Psi(p)$$
$$\langle x \mid p \rangle = (2\pi\hbar)^{-1/2} \exp(ipx/\hbar)$$
$$\langle p \mid \alpha \rangle = \int dx \langle p \mid x \rangle \langle x \mid \alpha \rangle$$

and thus obtain the following system of equations:

$$-\varphi(x) \ln \varphi^*(x)\varphi(x) - (2\pi\hbar)^{-1/2} \int_{-\infty}^{\infty} \exp(ipx/\hbar)\Psi(p) \ln \Psi^*(p)\Psi(p)\, dp = W\varphi(x) \quad (3.1a)$$

$$\Psi(p) = (2\pi\hbar)^{-1/2} \int_{-\infty}^{\infty} \exp(-ipx/\hbar)\varphi(x)\, dx \quad (3.1b)$$

Since these are difficult to solve we have found, experimentally, that all wave functions which minimize the standard uncertainty relation also exaggerate the entropic uncertainty relation, i.e. they are solutions of equations (3.1). These functions in x-representation are

$$\varphi(x) = (B/\pi)^{1/4} \exp[ir - (\operatorname{Re} A)^2/2B] \exp[-(B/2)x^2 + Ax] \quad (3.2a)$$

and in p-representation are

$$\Psi(p) = (B/\pi)^{1/4}(2\pi h)^{-1/2} \exp[ir - (\operatorname{Re} A)^2/2B] \exp(A - ip/h)^2$$
$$B > 0 \quad \operatorname{Im} r = 0 \quad (3.2b)$$

The fact of r being real not only follows from the normalisation condition but also from the non-linearity of equations (3.1). The value of the uncertainty sum (2.1) belonging to function (3.2) is

$$W = \ln \pi e\hbar \quad (3.3)$$

and is independent of the parameters A, B and r. Since we do not know whether the functions (3.2) represent all solutions of equations (3.1), we cannot determine whether they minimise W. In spite of this we hope that the uncertainty relation

$$W(X) + W(P) \geqslant \ln \pi e\hbar \quad (3.4)$$

is valid, at least for some class of the wave functions. In order to promote this supposition, we can show that there exist functions to which belong a larger value of W than that of equation (3.3). Such functions, for example, in the x-representation, are

$$\varphi(x) = a^{-1/2}\pi^{-1/4(x\,|\,a)} \exp(-x^2/2a^2) \tag{3.5a}$$

and in the p-representation

$$\Psi(p) = -i(a/\hbar)^{1/2}\pi^{-1/4}(pa/\hbar) \exp(-p^2s^2/2\hbar^2) \tag{3.5b}$$

where a is an arbitrary real non-negative parameter. To this function belongs the following value of W:

$$W = \tfrac{1}{2} \ln \pi\hbar + \tfrac{3}{2} + C + \ln 4 > \ln \pi\, e\hbar \tag{3.6}$$

where C is Euler's constant. In the calculations we have used (Grebner & Hofreiter, 1950, formulae 324, 83a and 411, 7c)

$$\int_0^\infty e^{-x}x^{u-1} \ln x\, dx = \Gamma(u)\varphi(u)$$

$$\varphi(1/2) = -C - \ln 4$$

We are not able to calculate exactly the integral of equation (1.4) of the entropy and the integral in equation (3.1a) even for the simplest quantum mechanical examples. Therefore we try to seek an approximation of the lower bound of W. To that purpose we use the generalised information energy

$$E(X) = \int_{(x)} [P(x)]^2\, dx \tag{3.7}$$

originally defined for discrete probability distributions (Vajda, 1967).

We obtain the following approximation

$$W = W(X) + W(P) = -\int |\varphi(x)|^2 \ln |\varphi(x)|^2\, dx - \int |\Psi(p)|^2 \ln |\Psi(p)|^2\, dp$$

$$= -\int |\varphi(x)|^2 |\Psi(p)|^2 \ln |\varphi(x)^2| \Psi(p)|^2$$

$$.\, dx\, dp \geqslant -\int |\varphi(x)|^4 |\Psi(p)|^4\, dx\, dp = -\int |\varphi(x)|^4\, dx\,.\int |\Psi(p)|^2\, dp$$

$$= -E(X)\,.\,E(P) = -E$$

$$W = W(X) + W(P) \geqslant -E(X)\,.\,(P) = -E \tag{3.8}$$

This approximation, of course, is only meaningful in the case of continuous probability distributions when the entropy can be negative. The minimum of $-E$ is then one of the approximations of the lower bound of W. As above, we use the variational method to determine the extremes of E. We obtain the following equations

$$E(P)\varphi^*(x)\varphi^2(x) + E(X)(2\pi\hbar)^{-1/2} \int_{-\infty}^{\infty} \exp(ipx/\hbar)\Psi^*(p)\Psi^2(p)\,dp$$
$$= 2E(X).R(P)\varphi(x) \qquad (3.9a)$$

$$\Psi(p) = (2\pi\hbar)^{-1/2} \int_{-\infty}^{\infty} \exp(-ipx/\hbar)\varphi(x)\,dx \qquad (3.9b)$$

Unfortunately the system (3.9) is again very complicated. We do not know its solution.

Since the harmonic oscillator eigenfunctions are of great significance in physics and also represent the stationary states of the standard uncertainty relation, we approximate the lower bound of their entropic sum by means of the approximation (3.8). We expect that E will increase and W will decrease with increasing n since the more the harmonic oscillator is excited the smoother is the wave packet. The wave functions of the harmonic oscillator, in the x-representation, are

$$\varphi_n(x) = (2^n n!\,\pi^{1/2}x_0)^{-1/2} H_n(x/x_0) \exp(-x^2/2x_0) \qquad (3.10a)$$

and in the p-representation

$$\Psi_n(p) = (-i)^n (2^n n!\,\pi^{1/2}p_0)^{-1/2} H_n(p/p_0) \exp(-p^2/2p_0) \qquad (3.10b)$$

where

$$x_0 p_0 = \hbar \qquad (3.10c)$$

must be valid. Information energies of the probability distribution are

$$E_n(X) = \int_{-\infty}^{\infty} |\varphi_n(x)^4\,dx = x_0^{-1} I_n \qquad (3.11a)$$

$$E_n(P) = \int_{-\infty}^{\infty} |\Psi_n(p)|^4\,dp = p_0^{-1} I_n \qquad (3.11b)$$

$$I_n = (2^n n!\,\pi^{1/2})^{-2} \int_{-\infty}^{\infty} [H_n(x)]^4 \exp(-2x^2)\,dx \qquad (3.12)$$

and the product E is

$$E_n = E_n(X).E_n(P) = \hbar^{-1}(I_n)^2 \qquad (3.13)$$

The value of the integral I_n (3.12) is (Appendix (A.6))

$$I_n = \frac{1}{(2\pi)^{1/2} n!} \left\{ \frac{d^n}{dt^n} [(1-t)^{-1/2} F(1/2, 1/2; 1; t)] \right\}_{t=0}$$

$$= \frac{1}{(2\pi)^{1/2} 2^{4n}} \sum_{k=0}^{n} \binom{2n-2k}{n-k}^2 \binom{2k}{k} 2^{2k} \qquad (3.14)$$

The values of I_n for $n = 0, 1, \ldots, 10$ can be found in Table 1. We shall now show that I_n and E_n, too, are bounded for all n. Let us denote

$$a(n) = \frac{\Gamma(+1/2)}{\Gamma(1/2)\Gamma(n+1)} \qquad (3.15)$$

TABLE 1. Values for $n = 1$ are approximate

n	$(2\pi)^{1/2} I_n$	hV_n
0	1	1
1	0·75	0·5625
2	0·6406	0·4104
3	0·5742	0·3297
4	0·5279	0·2787
5	0·4930	0·2430
6	0·4652	0·2164
7	0·4426	0·1959
8	0·4235	0·1794
9	0·4071	0·1657
10	0·3927	0·1542

Then

$$a(n+1) = \frac{n+1/2}{n+1} a(n) < a(n) \qquad (3.16)$$

$$a(O) = 1 \quad a(n+1) < a(n) \qquad n = 0, 1, \ldots$$

$$n \to \lim - a(n) = 0$$

The integral I_n can be written in terms of $a(n)$ as

$$(2\pi)^{1/2} I_n = \sum_{k=0}^{n} a^2(n-k) a(k) \leqslant \sum_{k=0}^{n} a(n-k) a(k) = S_n \qquad (3.17a)$$

We define function $S(t)$ for $|t| < 1$

$$S(t) = \sum_{n=0}^{\infty} S_n t^n = \sum_{n=0}^{\infty} \sum_{k=0}^{n} a(n-k)a(k)t^n = \left[\sum_{n=0}^{\infty} a(n)t^n\right]^2 = [(1-t)^{-1/2}]^2$$

$$= (1-t)^{-1} = \sum_{n=0}^{\infty} t^n \tag{3.17b}$$

From here it follows that $S_n = 1$ and from (3.17b) it follows that

$$0 \leqslant I_n \leqslant (2\pi)^{-1/2} \tag{3.18}$$

From equations (3.8) and (3.13) we obtain

$$0 \leqslant E_n = E_n(X) . E_n(P) \leqslant 1/\hbar$$
$$W(X) + W(P) \geqslant -1/\hbar \tag{3.19}$$

Thus it is shown that the sum of the entropies W of the pure states of the linear harmonic oscillator has lower bounds no less than $-1/h$. So there exists the set of an infinite number of functions satisfying the uncertainty relation

$$W(X) + W(P) \geqslant -1/h \tag{3.20}$$

4. *Entropic Uncertainty relation for Spin Variables*

As a second example the entropic uncertainty relation in the case of a discrete probability distribution is calculated. In this case the significance of the new formalism just excel because of the definition of the entropy. Let us consider a quantum mechanical system of spin $h/2$. The projections of the spin σ_z to the axes x and z of the cartesian coordinate system are represented by the operators

$$\hat{\sigma}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \hat{\sigma}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4.1}$$

The state vectors are the two-component spinors

$$|\Psi\rangle = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \tag{4.2}$$

normalised to unity

$$a_1^* a_1 + a_2^* a_2 = 1 \tag{4.3}$$

In order to stress the advantage of the entropic uncertainty relation, we shall first of all study the standard uncertainty relation. Let us, therefore, minimise the uncertainty product

$$U(a_1 . a_2) = \langle(\hat{\sigma}_x - \langle\hat{\sigma}_x\rangle)^2\rangle\langle(\hat{\sigma}_z - \langle\hat{\sigma}_z\rangle)^2\rangle \tag{4.4}$$

Using equations (4.1) and (4.2) we obtain

$$U(a_1 . a_2) = \frac{\hbar^4}{16}[1 - (a_1^* a_1 - a_2^* a_2)^2][1 - (a_1^* a_2 + a_1 a_2^*)^2] \tag{4.5}$$

We shall now introduce new variables (with respect to the normalisation condition (4.3))

$$a_1 = r\,e^{i\varphi_1} a_z = \sqrt{(1 - r^2)}\,e^{i\varphi_2} \varphi = \varphi_1 - \varphi_2 \tag{4.6}$$

and after substituting them into equation (4.5) we have

$$U(r, q) = \frac{\hbar^4}{4} r^2(1 - r^2)[1 - 4r^2(1 - r^2)\cos^2\varphi] \tag{4.7}$$

The necessary conditions for the extreme points of this function are as follows:

$$\frac{\partial U}{\partial\varphi} = 2\hbar^4 r^4(1 - r^2)^2 \cos\varphi \sin\varphi = 0 \tag{4.8a}$$

$$\frac{\partial U}{\partial r} = \frac{\hbar^4}{4} r(1 - 2r^2)[1 - 8r^2(1 - r^2)\cos^2\varphi] = 0 \tag{4.8b}$$

The physically allowed roots of equations (4.8) are

$$r = 0 \qquad \varphi \text{ arbitrary}$$

$$r = \sqrt{\left(\tfrac{1}{2} - \frac{\sqrt{2}}{4}\right)} \doteq 0{\cdot}382683\ \ldots\ \varphi = 0, \pi$$

$$r = 1/\sqrt{2} \qquad \varphi = 0, \pi/2, 3\pi/2 \tag{4.9a}$$

$$r = \sqrt{\left(\tfrac{1}{2} + \frac{\sqrt{2}}{4}\right)} \doteq 0{\cdot}923885\ \ldots\ \varphi = 0, \pi$$

Limit points must also be considered.

$$r = 1, \qquad \varphi \text{ arbitrary} \tag{4.9b}$$

The values of the uncertainty product (4.8) and the vectors belonging to the points (4.9) are

$$U(0, \varphi) = 0 \qquad\qquad |\Psi\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$U\left[\sqrt{\left(\tfrac{1}{2}-\frac{\sqrt{2}}{4}\right)},0\right]=\frac{\hbar^4}{64} \qquad |\Psi\rangle=\begin{bmatrix}\sqrt{\left(\tfrac{1}{2}-\frac{\sqrt{2}}{4}\right)}\\ \sqrt{\left(\tfrac{1}{2}+\frac{\sqrt{2}}{4}\right)}\end{bmatrix}$$

$$U\left[\sqrt{\left(\tfrac{1}{2}-\frac{\sqrt{2}}{4}\right)},\pi\right]=\frac{\hbar^4}{64} \qquad |\Psi\rangle=\begin{bmatrix}\sqrt{\left(\tfrac{1}{2}-\frac{\sqrt{2}}{4}\right)}\\ -\sqrt{\left(\tfrac{1}{2}+\frac{\sqrt{2}}{4}\right)}\end{bmatrix}$$

$$U[1/\sqrt{2},0]=0 \qquad |\Psi\rangle=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$$

$$U(1/\sqrt{2},\pi)=0 \qquad |\Psi\rangle=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$$

$$U(1/\sqrt{2},\pi/2)=\frac{\hbar^4}{16} \qquad |\Psi\rangle=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}$$

$$U(1/\sqrt{2},3\pi/2)=\frac{\hbar^4}{16} \qquad |\Psi\rangle=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}$$

$$U\left[\sqrt{\left(\tfrac{1}{2}+\frac{\sqrt{2}}{4}\right)},0\right]=\frac{\hbar^4}{64} \qquad |\Psi\rangle=\begin{bmatrix}\sqrt{\left(\tfrac{1}{2}+\frac{\sqrt{2}}{4}\right)}\\ \sqrt{\left(\tfrac{1}{2}-\frac{\sqrt{2}}{4}\right)}\end{bmatrix}$$

$$U\left[\sqrt{\left(\tfrac{1}{2}+\frac{\sqrt{2}}{4}\right)},\pi\right]=\frac{\hbar^4}{64} \qquad |\Psi\rangle=\begin{bmatrix}\sqrt{\left(\tfrac{1}{2}+\frac{\sqrt{2}}{4}\right)}\\ \sqrt{\left(\tfrac{1}{2}-\frac{\sqrt{2}}{4}\right)}\end{bmatrix}$$

$$U(1,\varphi)=0 \qquad |\Psi\rangle=\begin{pmatrix}1\\0\end{pmatrix} \tag{4.10}$$

Thus the following standard uncertainty relation is valid

$$0\leqslant\langle(\sigma_x-\langle\sigma_x\rangle)^2\rangle\langle(\sigma_z-\langle\sigma_z\rangle)^2\rangle\leqslant\hbar^4/16 \tag{4.11}$$

It attains its lower bound for eigenvectors of the operators σ_x and σ_z and its upper bound for eigenvectors of the operator σ_y. We see that in spite of the fact that there does not exist any vector which represents the eigenvector of both operators σ_x and σ_z the standard uncertainty relation can attain its zero minimum. This is a consequence of the fact that one dispersion is equal to zero whereas the other is non-zero and finite (as has already been mentioned in the introduction.) As we know, the entropic uncertainty relation is free from this defect; this will be shown in the following example.

Let

$$|\Psi\rangle = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \tag{4.12}$$

be a spinor in the σ_z-representation. We denote eigenvectors of the operator σ_z as follows

$$|z_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad |z_2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{4.13}$$

Thus the spinor (4.12) can be written in the form

$$|\Psi\rangle = a_1|z_1\rangle + a_2|z_2\rangle \tag{4.14}$$

Similarly, we denote eigenvectors of the operator σ_x as

$$|x_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad |x_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{4.15}$$

It is known from the theory of representations that

$$|z_i\rangle = \sum_j |x_j\rangle\langle x_j|z_i\rangle \qquad i, j = 1, 2 \tag{4.16}$$

Thus the spinor (4.12) is in the σ_x-representation

$$|\Psi\rangle = \sum_{i,j} a_i\langle x_j|z_i\rangle|x_j\rangle \tag{4.17}$$

where the transformation matrix is

$$\langle x_j|z_i\rangle = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \tag{4.18}$$

Finally, we have the concrete form of the spinor (4.12) in the σ_x-representation

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} a_1 + a_2 \\ a_1 - a_2 \end{pmatrix} \tag{4.19}$$

Thus we obtain for the sum of the entropies in both representations

$$
\begin{aligned}
W = W(\sigma_x) + W(\sigma_z) &= -a_1^* a_1 \ln a_1^* a_1 - a_2^* a_2 \ln a_2^* a_2 \\
&- \tfrac{1}{2}(a_1^* + a_2^*)(a_1 + a_2) \ln \tfrac{1}{2}(a_1^* + a_2^*)(a_1 + a_2) \\
&- \tfrac{1}{2}(a_1^* - a_2^*)(a_1 - a_2) \ln \tfrac{1}{2}(a_1^* - a_2^*)(a_1 - a_2)
\end{aligned} \tag{4.20}
$$

or, using the variables (4.6),

$$
\begin{aligned}
W = &-r^2 \ln r^2 - (1 - r^2) \ln (1 - r^2) - \tfrac{1}{2}(1 + 2r\sqrt{(1 - r^2)} \cos \varphi) \ln \tfrac{1}{2}(1 \\
&+ 2r\sqrt{(1 - r^2)} \cos \varphi) - \tfrac{1}{2}(1 - 2r\sqrt{(1 - r^2)} \cos \varphi) \ln \tfrac{1}{2}(1 \\
&- 2r\sqrt{(1 - r^2)} \cos \varphi)
\end{aligned} \tag{4.21}
$$

The necessary conditions for the extremes are

$$
\frac{\partial W}{\partial \varphi} = r\sqrt{(1 - r^2)} \sin \varphi \ln \frac{1 + 2r\sqrt{(1 - r^2)} \cos \varphi}{1 - 2r\sqrt{(1 - r^2)} \cos \varphi} = 0 \tag{4.22a}
$$

$$
\begin{aligned}
\frac{\partial W}{\partial r} = &- 2r \ln r^2 + 2r \ln (1 - r^2) - \frac{1 - 2r^2}{\sqrt{(1 - r^2)}} \cos \varphi \ln \frac{1 + 2r\sqrt{(1 - r^2)} \cos \varphi}{1 - 2r\sqrt{(1 - r^2)} \cos \varphi} \\
= &\ 0
\end{aligned} \tag{4.22b}
$$

The physically allowed roots of equations (4.22) are all points (4.9a) and, of course, limit points (4.9b) must also be considered. We have ascertained by use of the computer that there are no other physically allowed roots of equations (4.22). Substituting the values (4.9) to the state vectors (4.10) into (4.21) we obtain

$$
\begin{aligned}
&W(0, \varphi) = W(1, \varphi) = W(1/\sqrt{2}, 0) = W(1/\sqrt{2}, \pi) = \ln 2 \doteq 0{,}693147 \ldots \\
&W\left[\sqrt{\left(\tfrac{1}{2} - \tfrac{\sqrt{2}}{4}\right)}, 0\right] = W\left[\sqrt{\left(\tfrac{1}{2} - \tfrac{\sqrt{2}}{4}\right)}, \pi\right] = W\left[\sqrt{\left(\tfrac{1}{2} + \tfrac{\sqrt{2}}{4}\right)}, 0\right] \\
&\qquad = W\left[\sqrt{\left(\tfrac{1}{2} + \tfrac{\sqrt{2}}{4}\right)}, \pi\right] \\
&\qquad = \ln 8 - \frac{1}{\sqrt{2}} \ln (3 + \sqrt{2}) \doteq 1{,}029504 \ldots < 2 \ln 2 \\
&W(1/\sqrt{2}, \pi/2) = W(1/\sqrt{2}, 3\pi/2) = 2 \ln 2 \doteq 1{,}386295 \ldots
\end{aligned} \tag{4.23}
$$

Therefore the following entropic uncertainty relation is valid

$$
\ln 2 \leqslant W(\sigma_z) + W(\sigma_z) \leqslant 2 \ln 2 \tag{4.24}
$$

or, if in the definition of the entropy $\log_2$ is used instead of ln,

$$1 \leqslant W(\sigma_x) + W(\sigma_z) \leqslant 2 \tag{4.25}$$

The entropic uncertainty relation (4.24) as well as the standard one (4.11) attains its minimum for eigenvectors of the operators σ_x and σ_z and its maximum for the eigenvectors of the operator σ_y. The entropic uncertainty relation (4.24) is evidently more adequate than the entropic uncertainty relation for continuous probability distributions which only consist of the 'regularised parts' of the entropy. The entropic uncertainty relation (4.24) is based on the entropies of discrete probability distributions which can immediately be defined as a natural measure of the uncertainty.

5. *Conclusion*

The entropic uncertainty relation represents no generalisation of the standard relation but, in principle, a new formulation using the entropy as a natural measure of the uncertainty of probability distributions. Practical applications of the entropic uncertainty relation are, however, considerably difficult because of the mathematical complications and for the reason that the entropy is not usually used in experimental practice. On the other hand, the statistical dispersions are frequently used in experimental practice to estimate the accuracy of the experimental results, thus in practice one prefers the standard uncertainty relation to the entropic one. The main meaning of the entropic uncertainty relation can, perhaps, be seen in the theoretical field, e.g. in connection with information theory. From what has been said so far we come to the following conclusion:

In the case of continuous probability distributions (canonically conjugated observables in the Weyl sense) we met profound great mathematical complications. It was found that all states minimising standard uncertainty relations enhanced the entropic one. We do not know which sort this extreme is but we hope, according to a functional dependence of the state vectors, that the extreme could be minimal. It was also shown that the uncertainty sum W of the linear harmonic oscillator eigenstates is bound by a value not less than $-1/\hbar$. The case of the discrete distributions (spin observables of a system with spin $\hbar/2$) is considered more mathematically adequate than the continuous one according to the definition of the entropy. All the calculations were completed without any approximations. Contrary to the standard uncertainty relation the entropic one, besides the fact that it represents the uncertainty better, also has another advantage—it has not the standard relation's drawback of attaining the zero minimum.

We wish to mention an interesting fact: In our example of the spin observable the extremising states are the same for both standard and entropic uncertainty relations and extremes belonging to them are also of the same sort.

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Appendix

For calculation of the integral (3.12) we begin by expressing the product $H_p(x)H_q(x)$ as a linear combination of $H_k(x)$. For that purpose we use the principal function of the Hermite polynomials

$$\exp(-t^2 + 2t) = \sum_{n=0}^{\infty} H_n(x)t^n/n!$$

$$H_p(x)H_q(x) = \frac{d^p}{dr^p}\frac{d^q}{ds^q}\exp(-r^2 + 2rx)\exp(-s^2 + 2sx)\,|_{r=s=0}$$

$$= \frac{d^p}{dr^p}\frac{d^q}{ds^q}\exp(2rs)\exp(-(r+s)^2 + 2(r+s)x)\,|_{r=s=0}$$

$$= \frac{d^p}{dr^p}\frac{d^q}{ds^q} f(2rs)g(r+s)\,|_{r=s=0} = \frac{d^p}{dr^p}\sum_{k=0}^{q}\binom{q}{k}2^k r^k f^{(k)} g^{(q-k)}\,|_{r=s=0}$$

$$= \sum_{k=0}^{q}\sum_{l=0}^{p}\binom{q}{k}\binom{p}{l}\frac{2^k k! r^{k-1}}{\Gamma(k-1+1)}\frac{d^{p-1}}{dr^{p-1}}[f^{(k)}g^{(q-k)}]\,|_{r=s=0}$$

$$= \sum_{k=0}^{q}\sum_{l=0}^{p}\sum_{t=1}^{p-1}\binom{q}{k}\binom{p}{-l}\binom{p-1}{t}\frac{2^{\;+t}k! r^{k-1}s^t}{\Gamma(k-l+1)} f^{(k+t)}$$

$$. g^{(p+q-k-l-t)}|_{r=s=0} = \sum_{k=0}^{q}\sum_{l=0}^{p}\sum_{t=0}^{p-1}\binom{q}{k}\binom{p}{l}\binom{p-1}{t}\frac{2^{k+t}k!}{\Gamma(k-l+1)}$$

$$. H_{p+q-k-l-t}(x)\,\delta_{0,t}\,\delta_{0,k-1}$$

$$= \sum_{k=0}^{\min(q,p)}\binom{q}{k}\binom{p}{k}2^k k!\, H_{p+q-2k}(x) \tag{A.1}$$

If $n = p = q$, we obtain from (A.1)

$$[H_n(x)]^2 = \sum_{k=0}^{n} \binom{n}{k} \frac{2^{n-k} n!}{k!} H_{2k}(x) \tag{A.2}$$

where we have used

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

and have changed k by $n - k$. Inserting (A.2) into (3.12) we have

$$I_n = \frac{1}{\pi} \sum_{k=0}^{n} \sum_{l=0}^{n} \binom{n}{k}\binom{n}{l} \frac{2-k-l}{k!l!} \int_{-\infty}^{\infty} H_{2k}(x) H_{2l}(x) \exp(-2x^2)\, dx$$

$$= \frac{1}{\pi} \sum_{k=0}^{n} \sum_{l=0}^{n} \binom{n}{k}\binom{n}{l} \frac{2-k-l}{k!l!} \int_{-\infty}^{\infty} H_{2k+2l} \exp(-2x^2)\, dx \tag{A.3}$$

Further, by use of the principal function of Hermitte polynomials we find that

$$\int_{-\infty}^{\infty} H_{2k+1} 2(x) \exp(-2x^2)\, dx$$

$$= \left[\frac{d^{2k+2l}}{dt^{2k+2l}} \int_{-\infty}^{\infty} \exp(-2x^2 - t^2 + 2tx)\, dt \right]_{t=0}$$

$$= \left(\frac{\pi}{2}\right)^{1/2} \left[\frac{d^{2k+2l}}{dt^{2k+2l}} \exp(-t^2/2) \right]_{t=0} = \left(\frac{\pi}{2}\right)^{1/2} (-1)^{k+1} \frac{(2k+2l)!}{(k+1)!}$$

$$= (-1)^{k+2} 2^{k+l-1/2} \Gamma(k + l + 1/2) \tag{A.4}$$

After inserting (A.4) into (A.3) and using

$$(k + l + 1/2) = \int_{0}^{\infty} \exp(-x_x k + l - 1/2)\, dx$$

the integral (3.12) takes the form

$$I_n = \frac{1}{\pi 2^{1/2}} \sum_{k=0}^{n} \sum_{l=0}^{n} \binom{n}{k}\binom{n}{l} \frac{(-1)^{k+l/1}}{k!l!} \int_0^\infty \exp(-x_x k + l - 1/2)\, dx$$

$$= \frac{1}{\pi 2^{1/2}} \int_0^\infty x^{-1/2} \exp(-x) \left[\frac{1^x}{n!} \frac{d^n}{dx^n} (x^n \mathrm{e}^{-x})\right]^2 dx$$

$$= \frac{1}{\pi 2^{1/2}} \int_0^\infty x^{-1/2} \exp(-x[L_n(x)]^2\, dx) \tag{A.5}$$

The integral (A.5) is a special case of the integral (Gradstein & Ryshik, 1963, formula 7, 414, 12)

$$\int_0^\infty \exp\left[-x\left(s + \frac{a_1 + a_2}{2}\right) x^{\tau+\beta} L_k^{\tau}(a_1 x) L_k^{\tau}(a_2 x)\, dx\right]$$

$$= \frac{\Gamma(1 + \tau + k)\Gamma(1 + \tau + \beta)}{(1 + \tau)k!k!}$$

$$\times \left\{ \frac{d^k}{dh^k} \left[\frac{F\left(\frac{1 + \tau + \beta}{2}, 1 + \frac{\tau + \beta}{2}; 1 + \tau; A^2/B^2\right)}{(1 - h)^{1+\tau} B^{1+\tau+\beta}} \right] \right\} h = 0$$

$$A^2 = \frac{4a_1 a_2 h}{(1 - h)^2} \qquad B = S + \frac{a_1 + a_2}{2} \frac{1 + h}{1 - h}$$

$$\mathrm{Re}\left(s + \frac{a_1 + a_2}{2}\right) > 0, \qquad a_1 > 0, \qquad a_2 > 0\ \mathrm{Re}(\tau + \beta) > -1$$

if $a_1 = a_2 = 1$, $s = 0$, $\tau = 0$, $\beta = -1/2$. If with the last integral we also use the formula (Gradstein & Ryshik, 1963, formulae 9, 134, 2)

$$F(2a, 2a + 1 - c; c; x) = (1 - x)^{-2a} F\left(a_1 a + \tfrac{1}{2}; c; \frac{4}{(1 - x)^2}\right)$$

we obtain

$$I_n = \frac{1}{(2\pi)^{1/2}} \left\{ \frac{d^n}{dx^n} [(1-x)^{-1/2} F(1/2, 1/2; 1; x)] \right\}_{x=0}$$

$$= \frac{1}{(2\pi)^{1/2} n!} \sum_{k=0}^{n} \binom{n}{k} \frac{\Gamma(n-k+1/2)}{\Gamma(1/2)} \frac{\Gamma(k+1/2)}{\Gamma(1/2)} \frac{\Gamma(k+1/2)}{\Gamma(1/2)k!}$$

$$= \frac{1}{(2\pi)^{1/2}} \sum_{k=0}^{n} \binom{2n-2k}{n-k} \binom{2k}{k}^2 2^{-2n-2k}$$

$$= \frac{1}{(2\pi)^{1/2} 2^{4n}} \sum_{k=0}^{n} \binom{2n-2k}{n-k}^2 \binom{2k}{k} 2^{2k} \tag{A.6}$$

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